# USE OF A COORDINATE TRANSFORMATION IN THE INCREMENTAL PHASE PLANE*

D. P. Lindorff**, Senior Member, IEEE

Summary

A co-ordinate transformation in the incremental phase plane is introduced which facilitates the design of nonlinear sampled-data control systems. Use of these coordinates permits the slope of a boundary specified by the nonlinearity to be directly related to a simple gain term in the transformed system. A realizable form of compensation is shown to evolve in terms of the system variables. By using the transformed coordinates it is possible to identify within each sectionally linear region of the phase plane a unique equation for the isoclines. Examples are presented to illustrate the method. These involve contactor, saturation and quantization type nonlinearities.

## I. INTRODUCTION

Sampled-data feedback systems containing a simple nonlinearity have been the subject of a number of investigations. Methods of analysis have been employed using both the describing function and the phase plane. This paper is concerned with the further development of phase-plane techniques.

In the literature, both continuous and discrete variables have been used as phase-plane coordinates. In order to facilitate the analysis of a contactor system, Izawa and Weaver have replaced the sampling action by a random transport lag.[1] This method of analysis is limited to contactor-type nonlinearities. Mullin and Jury have adapted the phase-plane techniques applicable to continuous systems to sampled-data systems.[2] However, the computational difficulties resulting from the need to identify time

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**Electrical Engineering Department, University of Connecticut, Storrs, Conn.

with the trajectories imposes a detraction to the method. Aseltine and Nesbit have suggested the incremental phase plane as a natural medium for studying the behavior of nonlinear sampled-data systems.[3] This paper is concerned with the more effective utilization of the incremental phase plane as a design vehicle.

When the nonlinearity in Figure 1 is sectionally linear, it is possible to define boundaries in the phase plane within which the control system behaves linearly. Compensation schemes may then be employed which produce a desired rotation or translation of these boundaries. However, in applying this technique to sampled-data systems, certain complications arise when the solution path crosses such a boundary. The origin of this difficulty is related to the presence of the finite zero in the pulse transfer function of the quadratic plant. It will be shown that this difficulty can be overcome by introducing a new set of coordinates in terms of which the solution path encounters no complication in traversing a boundary. As a result, the problems of analysis and design can be significantly simplified.

In the development to follow, the concept of isoclines as applied to the incremental phase plane will be discussed briefly since this constitutes the computational tool to be employed throughout the paper. The new phase-plane coordinates will then be introduced, and attention will be directed to the mapping of boundaries and system variables of interest into this coordinate system. The form of compensation derived in the new coordinate system will be shown to be physically realizable when related to system variables. Finally, several examples will be discussed for the purpose of illustrating the technique advanced in the paper and providing some insight to the type of behavior which results from the presence of various forms of simple nonlinearities in sampled-data systems. These shall include the contactor, saturation and quantization.

## II. Use of Isoclines in the Incremental Phase Plane

The coordinates of the incremental phase plane (henceforth to be called the phase plane) will be defined in terms of the variable $x$ and its first forward difference, $\Delta x = x(n+1) - x(n)$, sometimes to be denoted by $y = \Delta x$. The solution is now sought to the second-order linear difference equation

$$\Delta^2 x + b\Delta x + cx = dm \tag{1}$$

where m is assumed constant. This solution can be represented by the locus of a set of points. Although the concept of a continuous trajectory has no meaning in the incremental phase plane, it is convenient to join successive solution points by straight-line segments as indicated in Figure 2. These connected line segments will be called a solution path of (1).

An isocline can now be defined in the $x, y$ plane in terms of the slope of these line segments. For each solution point $p_i$ which is nonsingular there is a slope $\Delta y/\Delta x$ associated with the line segment connecting $p_i$ with the succeeding solution point $p_{i+1}$. The locus of points which <u>initiates</u> line segments of constant slope is defined as an isocline. Thus with $k = \Delta y/\Delta x$, the isoclines are found from (1) to be given by the equation

$$k = \frac{dm}{y} - \frac{cx}{y} - b \tag{2}$$

For example, in Figure 2 the isocline passing through $p_1$ is described by (2) with $k =$ [illegible]. It can be seen from these statements that a point $p_2$ can be determined from a knowledge of $p_1$ and its isocline.

Difficulty is encountered in applying the method of isoclines when the solution point lies on the $x$ axis since the resulting solution is of indeterminant form. However, since $\Delta x = 0$ if $k$ is infinite, the increment in $y$ can be obtained directly from the difference equation. Thus if a solution point $p_1$ is located at $x = x_1$, and $y = 0$, then from (1) the subsequent value of $y$ becomes $y_2 = \Delta y = dm - cx_1$.

## III. Introduction of a New Set of Coordinates

In order to establish a motivation for introducing a new set of coordinates, consider the problem which is encountered in plotting a solution path for the system in Figure 1 when using the system variables $e$, $\Delta e$ as the phase-plane coordinates. With $r = 0$, the difference equation for the plant can be written in the form

$$\Delta e + b\Delta e + ce = dm + f\Delta m \qquad (3)$$

If the nonlinearity is assumed to be sectionally linear, it appears that the forcing function in (3) will not in general fit the form of (1). In particular, each time the solution path crosses a boundary established by the nonlinearity, a term in $\Delta m$ will be introduced. Elimination of the zero in the plant transfer function would apparently remedy this problem, in which case the form represented by (1) and (2) would apply throughout the region of the phase plane. As a means of attaining this objective consider the system in Figure 3. For the present let $K_c = 1$. In so far as $u$ is concerned it will be seen that this system is equivalent to the system in Figure 1 when the compensation has been deleted (The initial condition $c(0)$ will be discussed in detail subsequently). For this special case ($K_c = 1$), it is noted that $e = u$. In particular, the variable $x$ has been defined so that the parallel feedback path generates a forward first difference, $\Delta x$. If the phase-plane coordinates are now taken to be $x, \Delta x$, then the following equation pertains:

$$e = u = x + \frac{\Delta x}{1 - \beta} \quad , \quad \text{for } K_c = 1. \tag{4}$$

In Figure 4, the above equation is plotted for values of $e = u =$ constant. Suppose for purposes of clarification that the nonlinearity is an ideal contactor without deadzone. Then the line $u = 0$ would constitute a switching boundary whose slope is determined by the gain term $1/(1 - \beta)$.

Since the slope of the lines $u =$ constant is controlled by the gain of the inner feedback path, a rotation of these lines can be accomplished by introducing a gain $K_c$ as suggested in Figure 3. It is observed, however, that with $K_c \neq 1$, $e$ and $u$ are no longer equivalent. In fact the following results obtain:

$$\left.\begin{aligned} u &= x + \frac{K_c \Delta x}{1 - \beta} \\ e &= x + \frac{\Delta x}{1 - \beta} \end{aligned}\right\} \tag{5}$$

From (5) it is seen that lines of constant $e$ bear a fixed relationship to the $x, y$ plane whereas the slope of the boundaries defined by constant $u$ are dependent upon $K_c$.

If values of $K_c$ other than unity are to be specified it is necessary to determine the compensation in Figure 1 which satisfies an arbitrary choice of $K_c$. It is readily shown that the compensation in Figure 5 meets this requirement. It should be noted that this compensation in the feedback path is physically realizable. In the event that the compensation is to be realized by a digital filter, the customary restriction should be imposed that the zero at $z = \beta$ must be located within the unit circle. In the event that the plant contains at least a single pole at $z = 1$, it can be seen that the compensation is realizable directly by sampling the first derivative of the response variable in which case the former restriction on the location of the zero no longer applies since cancellation is not involved. This statement can be clarified by referring to the diagram of a quadratic plant in Figure 6 which contains a single pole at $S = 0$. The pulse transfer function $C/M$ for this system is of the form

$$\frac{C}{M} = \frac{K_1 (z - \beta)}{(z - 1)(z - \alpha_1)}$$

If the output rate is now considered as an output, the transfer function would become

$$\frac{\dot{C}}{M} = \frac{K_2}{(z - \alpha_1)}$$

By identifying the above equations with Figure 5, with $\alpha_2 = 1$, it will be seen that the compensating filter indicated in the diagram generates a signal which is proportional to the sampled values of $\dot{c}$.

Thus far, reference has been made to the system exclusively in terms of the pulse transfer function. In order now to establish certain relationships between the physical problem and the mathematical model used in the analysis, it is required that reference be made to the continuous-system variables, representative components of which appear in Figure 6.

One point of fundamental importance should not be overlooked. In the original system configuration of Figure 1 it was tacitly assumed that the nonlinearity was located as shown in Figure 6. This was required in order that the linear pulse transfer function could be defined for the plant and data-hold combination. If the nonlinearity possesses zero memory, however, it is permisseable to interchange the nonlinearity and the data-hold; conversely, a nonlinearity with memory is not interchangeable. The import of this statement is that a contactor with hysteresis cannot be treated by the methods presented in this paper. The reader is referred to the literature for discussion of hysteresis effects in sampled-data systems.[1,4]

Attention is now directed to the manner in which the initial conditions, $C(0)$ and $\dot{c}(0)$, are related to the $x, y$ plane. It will be shown that the discrete variable $x$ can be generated by sampling the signals derived from $c$ and $\dot{c}$ as indicated in Figure 6. The method of determining the proper values of $K_4$ and $K_5$ will be to equate $X/M$ as derived from Figure 6 to an equivalent expression derived from Figure 3.

From Figure 6 it follows that

$$\frac{X}{M} = \frac{(1-\alpha_1)K_4}{(z-\alpha_1)} - \frac{(1-\alpha_1)}{(1-\beta)} \frac{T(z-\beta)K_5}{(z-1)(z-\alpha_1)} \tag{6}$$

where T equals the sample period, $\alpha_1 = e^{-aT}$, and $\beta$ defines the zero of the pulse transfer function of the plant-hold combination. From Figure 3 with $\alpha_2 = 1$ it follows that

$$\frac{X}{M} = - \frac{T(1-\alpha_1)}{(z-\alpha_1)(z-1)} \tag{7}$$

where $\alpha_1$, $\beta$ and $T$ are as defined above. Equating (6) and (7), it is easily shown that

$$\left.\begin{aligned} K_4 &= \frac{T}{(1-\beta)} \\ K_5 &= 1 \end{aligned}\right\} \tag{8}$$

These results are still valid if the plant contains a second-order pole at $s=0$ in which case $\beta = -1$. When the plant contains poles at $-a_1$, and $-a_2$, i.e. no poles at $s=0$, the result becomes

$$\left.\begin{aligned} K_4 &= \frac{T}{1-\beta} \frac{(1-\alpha_1)(1-\alpha_2)(a_2-a_1)}{a_1 a_2(\alpha_1-\alpha_2)} \\ K_5 &= 1 \end{aligned}\right\} \tag{9}$$

where $\alpha_1 = e^{-a_1 T}$, $\alpha_2 = e^{-a_2 T}$, and $\beta$ and $T$ have their former meaning.

These results permit initial conditions in $x$ and $y$ to be derived from an arbitrary set of conditions $c(0)$, $\dot{c}(0)$ . Thus from Figure 6

$$x(0) = -c(0) + K_4 \dot{c}(0) \tag{10}$$

and from (5) with $e(0) = -c(0)$

$$\frac{y(0)}{1-\beta} = -c(0) - x(0) \tag{11}$$

From (10) and (11) it follows that

$$\left.\begin{aligned} y(0) &= -(1-\beta) K_4 \dot{c}(0) \\ x(0) &= -c(0) - \frac{y(0)}{1-\beta} \end{aligned}\right\} \tag{12}$$

## IV. EXAMPLES

### A. Contactor

The system to be considered is shown in Fig. 5, the transformed configuration being shown in Fig. 3. The nonlinearity to be used is described in Fig. 7. In this example the purpose is to illustrate an application of the method and illustrate how variations in $K_c$ affect the periodic state of the system. In the specific case to be treated it is assumed that $\alpha_1 = 0.367$, $\alpha_2 = 1$, $\beta = -0.72$, $K_1 = 1$ and the dead zone is a small quantity. From Figure 3 the pertinent difference equation then becomes

$$-(1-\beta)m = \Delta^2 x + (1-\alpha_1)\Delta x \tag{13}$$

This equation is seen to be in the form of (1). According to (2), the isoclines are defined by the equation

$$k = -(1-\beta)\frac{m}{y} - (1-\alpha_1) \tag{14}$$

The remaining equations of interest are given by (5). For the uncompensated system ($K_c = 1$), the lines of constant $e$ and $u$ are equivalent. The switching boundary for this case is plotted in Figure 8, together with a solution path starting at $x = -c(0) = -2.5$. It is observed that a limit cycle is formed with approximately unit amplitude of the error variable.

Consider now the effect of $K_c$ in Figure 3. From (5) it is perceived that an increase in $K_c$ ($K_c > 1$) will cause a counterclockwise rotation of the lines $u =$ constant. (Note that the lines of $e =$ constant are unaffected.) The periodic state of the system will now be investigated for the condition shown in Figure 9 in which $K_c$ has been adjusted so that the switching boundary passes below the solution point (1). The solution path can again be computed using (13) and (14). The amplitude of the resulting limit cycle, measured in terms of the error variable, is seen to be considerably reduced in amplitude. It is of interest to note that the bias upon which the limit cycle is superposed is a function of the initial condition, $c(0)$. Thus a change in $c(0)$ will shift the position of the limit cycle along the $x$ axis. Since the isoclines given by (14) are a function of $y$ only, the limit cycle itself is not dependent upon $x$ (except in so far as the requirement for alternating values of $m$ is satisfied by the slope of the switching line).

The results which have been derived above are intended to suggest the manner in which the periodic state of the contactor system can be altered by means of the compensating gain $K_c$. Although the discussion has been limited to initial states in which $\dot{c}(0) = 0$, the method may be applied to an arbitrary set of initial conditions.

B. Saturation

In this example the way in which system bandwidth affects the saturation problem is to be investigated. The system to be considered is represented in Figure 10A, the transformed system being shown in Figure Figure 10B. At this point it will be assumed that the transmission of the quantizer can be approximated by $Q = 1$. The approach to be taken will be first to select $K_c$ and $K_1$ so as to obtain the desired roots of the characteristic equation, assuming linear operation, and then to analyze the effect of saturation, using the transformed coordinates. For convenience the bandwidth will be adjusted under the constraint that the poles are critically damped. Accordingly two cases will be treated for which the characteristic equation is of the form $(z - \gamma)^2 = 0$. The corresponding linear homogeneous difference equation is given by $\Delta^2 x + b\Delta x + c x = 0$.

A specific solution will now be obtained by arbitrarily assigning the numerical values $T = 1$, $L = 1$ to the system parameters in Figure 10. If the roots of the characteristic equation at $z = \gamma$ are specified to represent a small and a large bandwidth, relatively speaking, the remaining parameters can be determined, as summarized in Table I. Referring to (2) and substituting the appropriate values from Table I, the isoclines for the unsaturated system are defined by the equations

$$\left.\begin{aligned} k &= -\frac{x}{4y} - 1, \quad \gamma = 0.5 \\ k &= -\frac{x}{y} - 2, \quad \gamma = 0 \end{aligned}\right\} \qquad (15)$$

The equation of the isoclines for the saturated system is common to both cases being treated, and is given by $k = -1/y$. Finally, the equation for the boundaries of the linear range in the $x, y$ plane is seen from the system in Figure 10B to be given by

$$\pm \frac{1}{K_1} = -x - \frac{K_c}{2} y \tag{16}$$

By applying the above equations, solution paths have been plotted in Figures 11 and 12 for the two cases being studied. In each case a solution path is identified which intersects the eigenvector at the saturation boundary. A comparison of these solutions indicates the degree to which the respective system designs are susceptible to saturation effects.

C. Quantization

Referring again to the system in Figure 10, the effect of quantization will be analyzed, assuming that the quantized inverval $q$ is small compared to the saturation level. Using the function for $Q$ which is graphically represented in Figure 13, the boundary lines in the $x, y$ plane are defined by

$$\frac{(1+2n)q}{2} = -x - \frac{K_c y}{2} \tag{17}$$

where $n$ is a positive or negative integer. The isoclines within a given strip are in turn defined by

$$k = \frac{K_1 n q}{y} \tag{18}$$

Using the values of $K_1$ and $K_c$ given in Table I, typical solution paths are plotted in Figures 13 and 14.

The results in Figure 14 warrant some discussion in that a limit cycle is indicated in the $x, y$ plane. This limit cycle differs from that of Figure 9, however, in the sense that it oscillates along a line of constant $e$, indicating that the sampled values of $e$ reach a steady value. Observation of Figure 6 shows that, as a consequence of the fact that $e(k) = -c(k) = \text{const.}$ it is necessary for $\dot{c}$ to vary in accordance with the periodicity of $x$. This state of affairs can exist only if an oscillation exists at one half the sampling frequency in the form of intersample ripple. The point to be made here is that an intersample ripple which would be undetectable in the $e, \Delta e$ plane is detectable in the $x, y$ plane in the form of a limit cycle along a line of constant $e$.

## V. CONCLUSIONS

A set of coordinates in the incremental phase plane has been introduced which permits the slope of a boundary specified by the nonlinearity to be directly related to a simple gain term in the transformed system. This gain in turn is equivalent to a simple form of realizable compensation in terms of the system variables. By using the new set of coordinates it is possible to identify within each sectionally linear region a unique equation for the isoclines. As a result, the concept of boundaries can be exploited in a design context.

Although in the analysis the nonlinearity is assumed to precede the data-hold, it is permissable to include nonlinearities at the output of the data-hold providing that they do not involve memory as in the case of hysteresis.

Examples involving a contactor, saturation and quantization have been presented to illustrate the manner in which a design can be carried out in the transformed coordinates.

It is felt that possibilities exist for extending the method to the problem of synthesizing switching boundaries for quasi-optimum control of sampled-data systems.

## VI. Bibliography


1. K. Izawa and L. E. Weaver, "Relay-type Feedback Control Systems with Dead Time and Sampling," Trans. AIEE, Vol. 78, Pt. II, pp. 49-54; 1959.

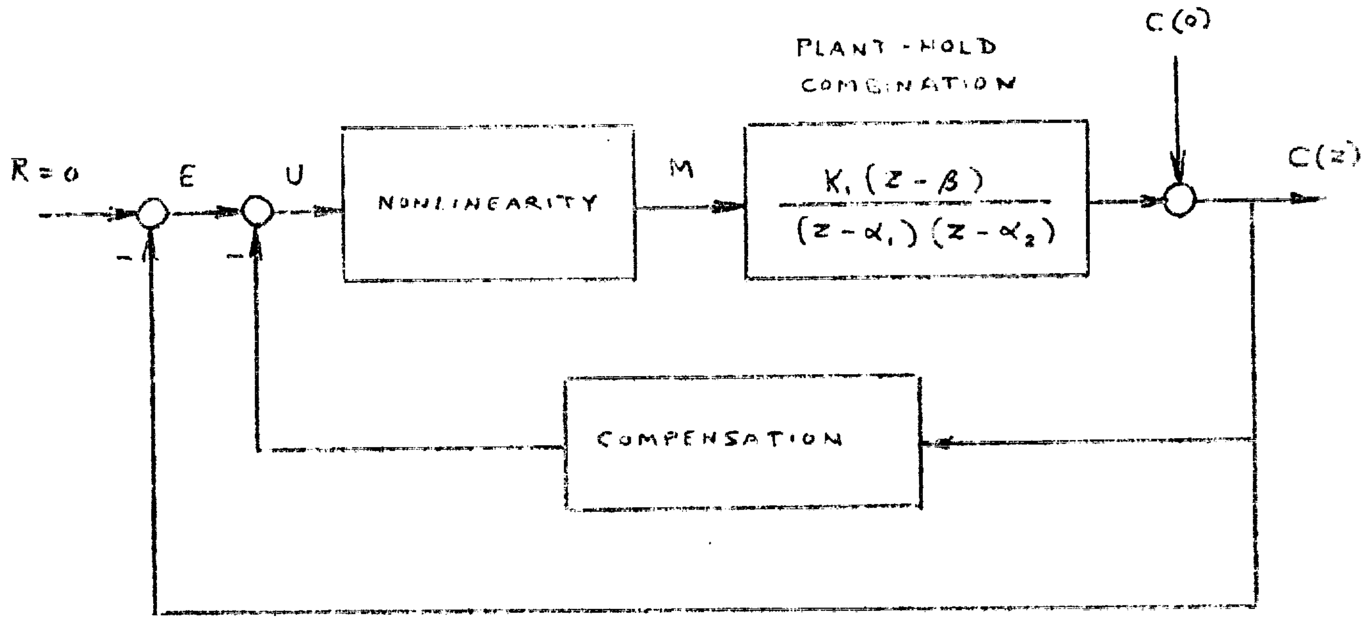
2. F. J. Mullin and E. I. Jury, "A Phase Plane Approach to Relay Sampled-Data Feedback Systems," Trans. AIEE, Vol. 78, Pt. II, pp. 517-524; 1959.

3. J. A. Aseltine and R. A. Nesbit, "The Incremental Phase Plane for Nonlinear Sampled Data Systems," IRE Convention Record, Pt. 4; 1960.

4. C. K. Chow, "Contactor Servomechanisms Employing Sampled Data," Trans. AIEE, Vol. 73, Pt. II, pp. 51-64; 1954.

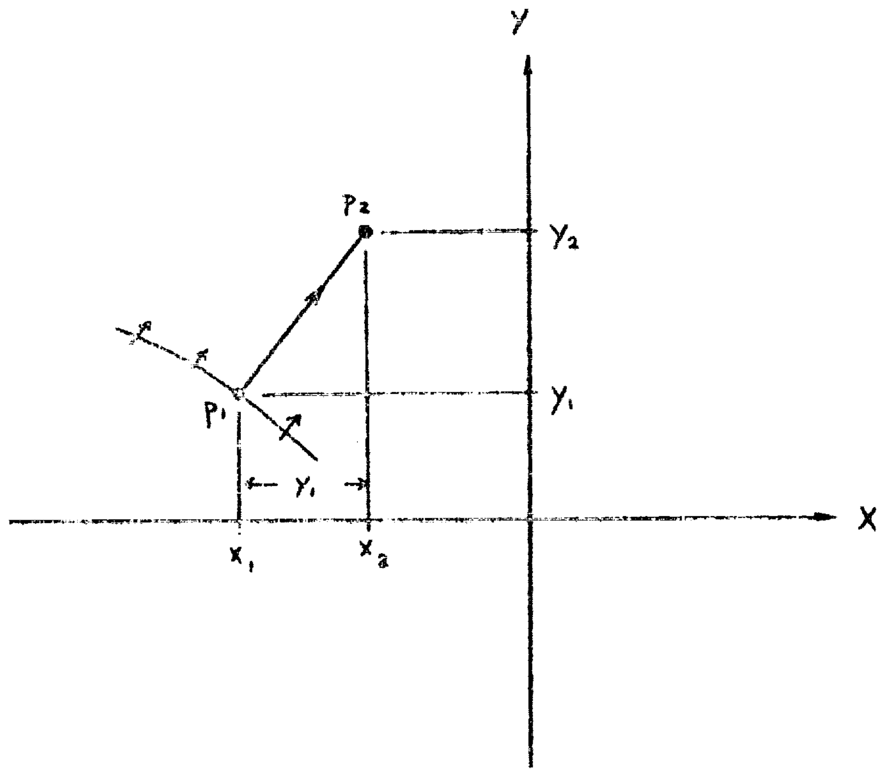
| Parameters | Small Bandwidth | Large Bandwidth |
| --- | --- | --- |
| | $\gamma = 0.5$ | $\gamma = 0$ |
| b | 1 | 2 |
| c | $\left(\frac{b}{2}\right)^2 = \frac{1}{4}$ | $\left(\frac{b}{2}\right)^2 = 1$ |
| $K_1$ | $\frac{1}{4}$ | 1 |
| $K_c$ | 8 | 4 |

Table 1



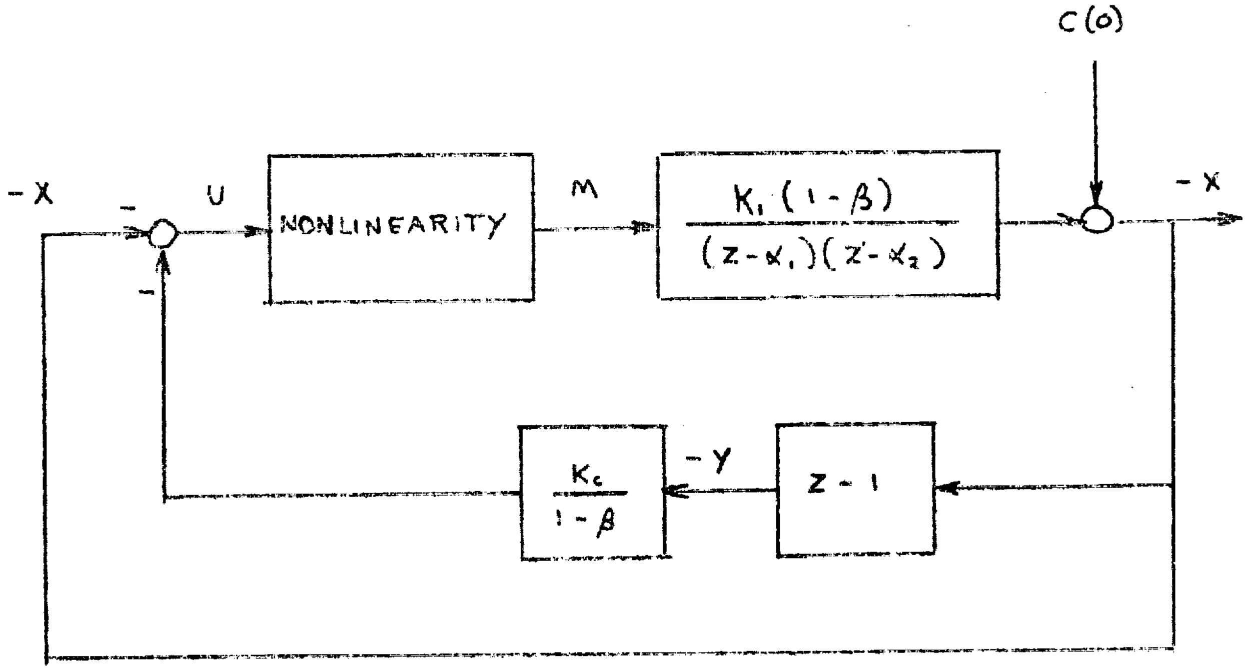

Basic Configuration—Quadratic System

FIGURE 1



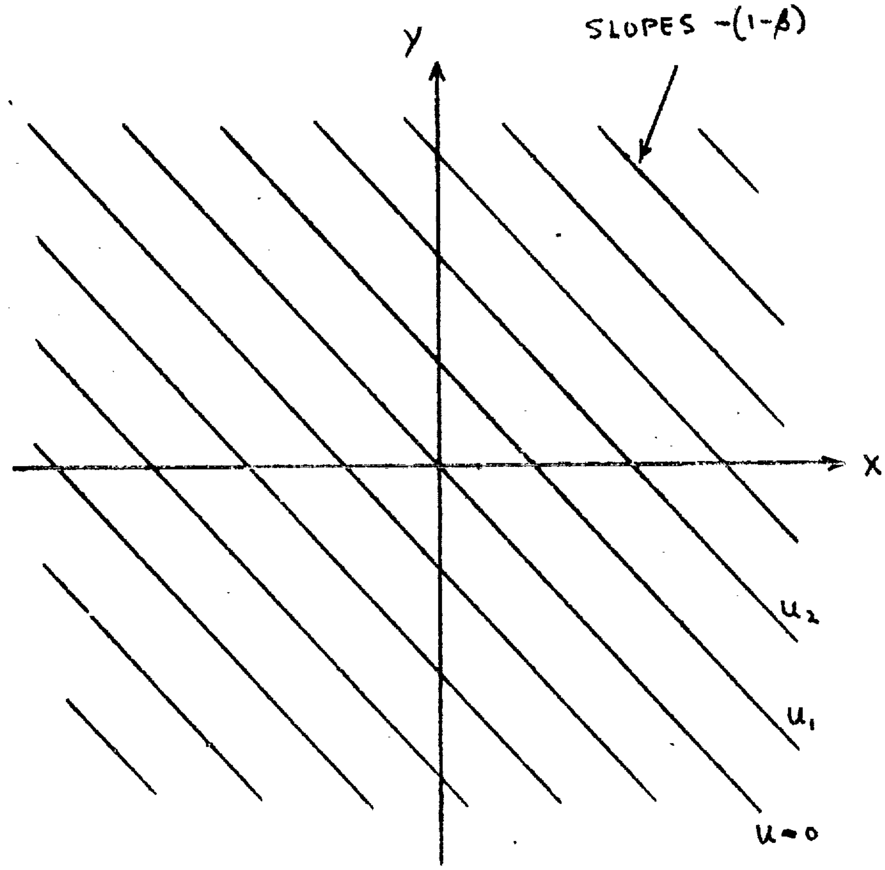

Example of Isocline

FIGURE 2



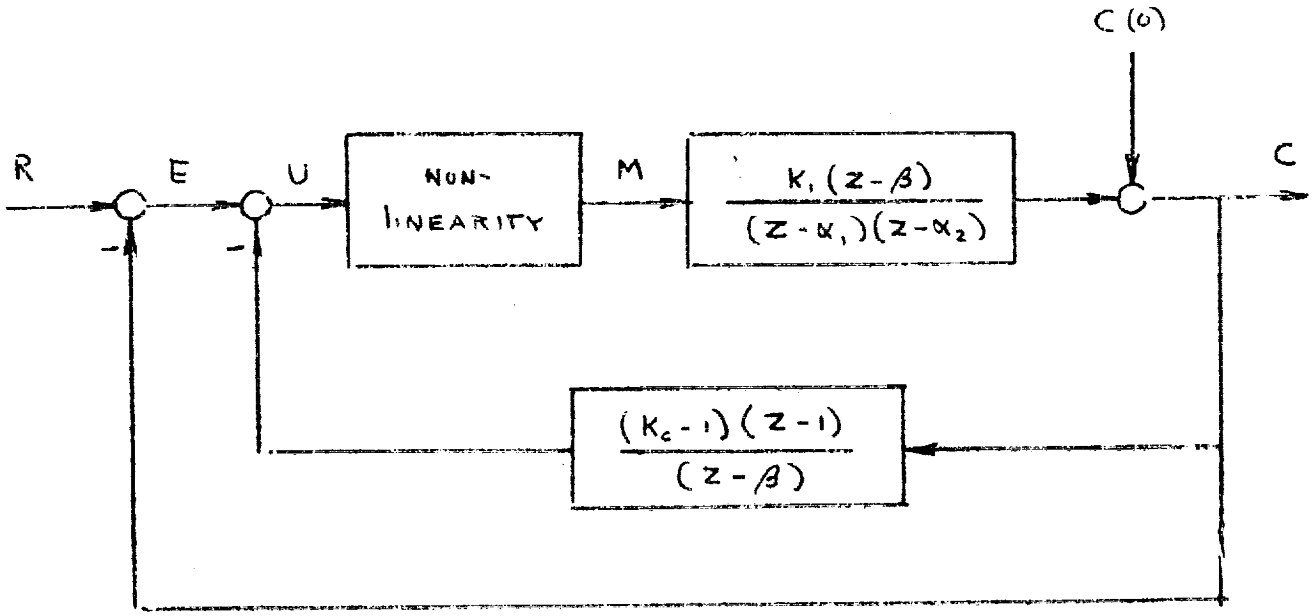

System Expressed in Terms of New Coordinates

FIGURE 3



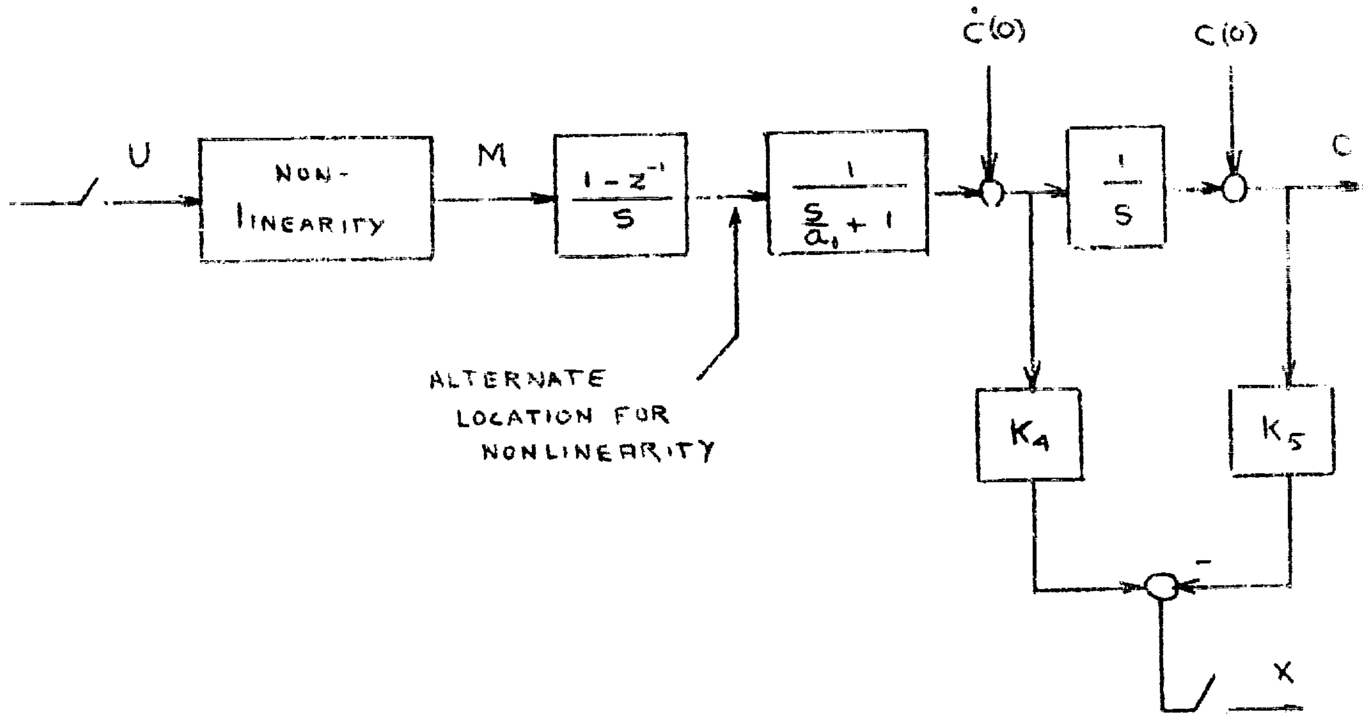

**Loci of constant U in xy plane**

FIGURE 4



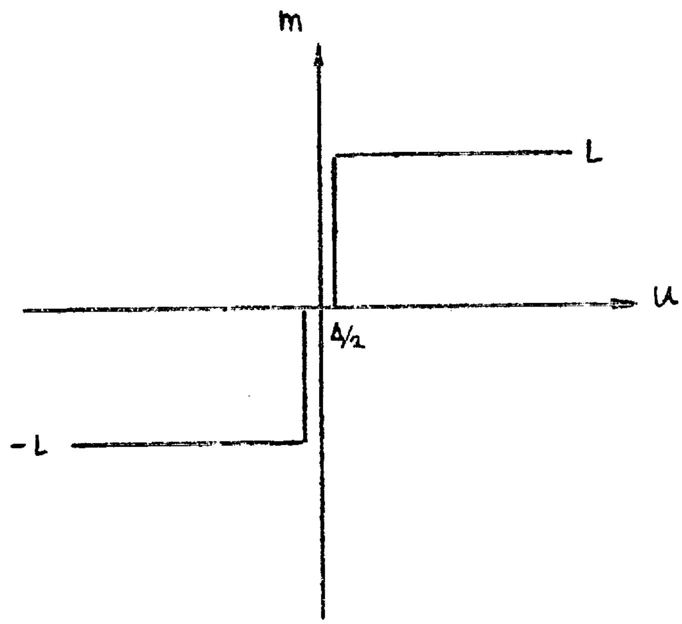

Compensated System

FIGURE 5



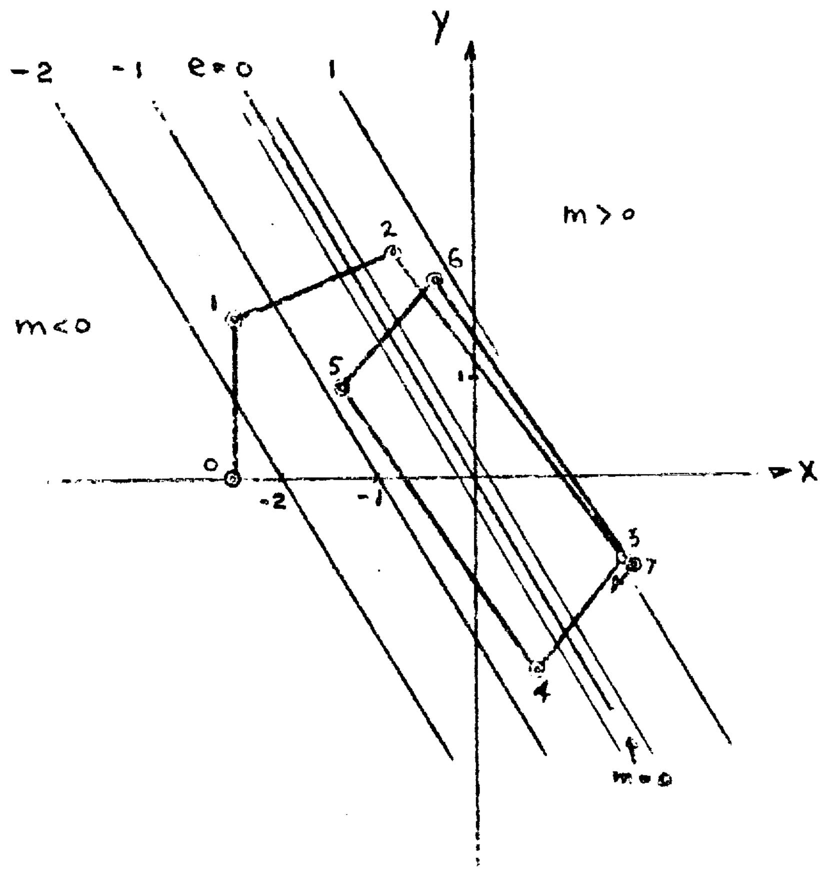

Generation of X from initial conditions

FIGURE 6



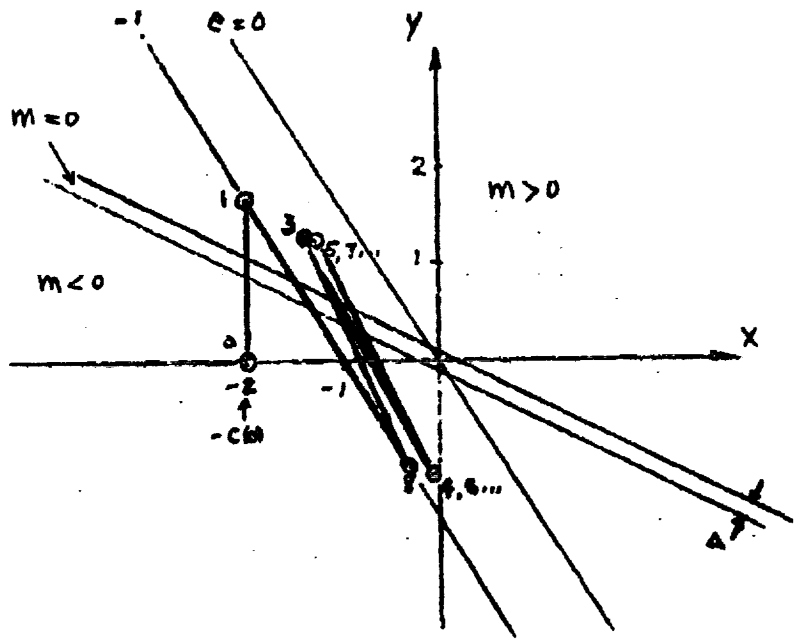

Contactor Nonlinearity

FIGURE 7



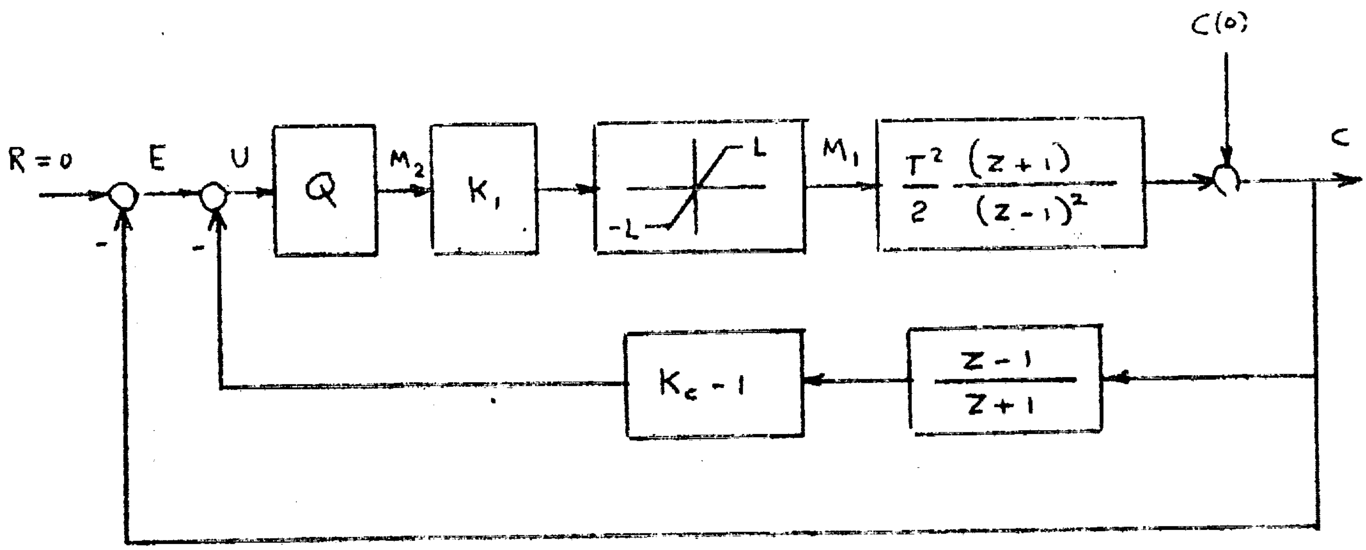

Contactor System Response—Uncompensated

FIGURE 8



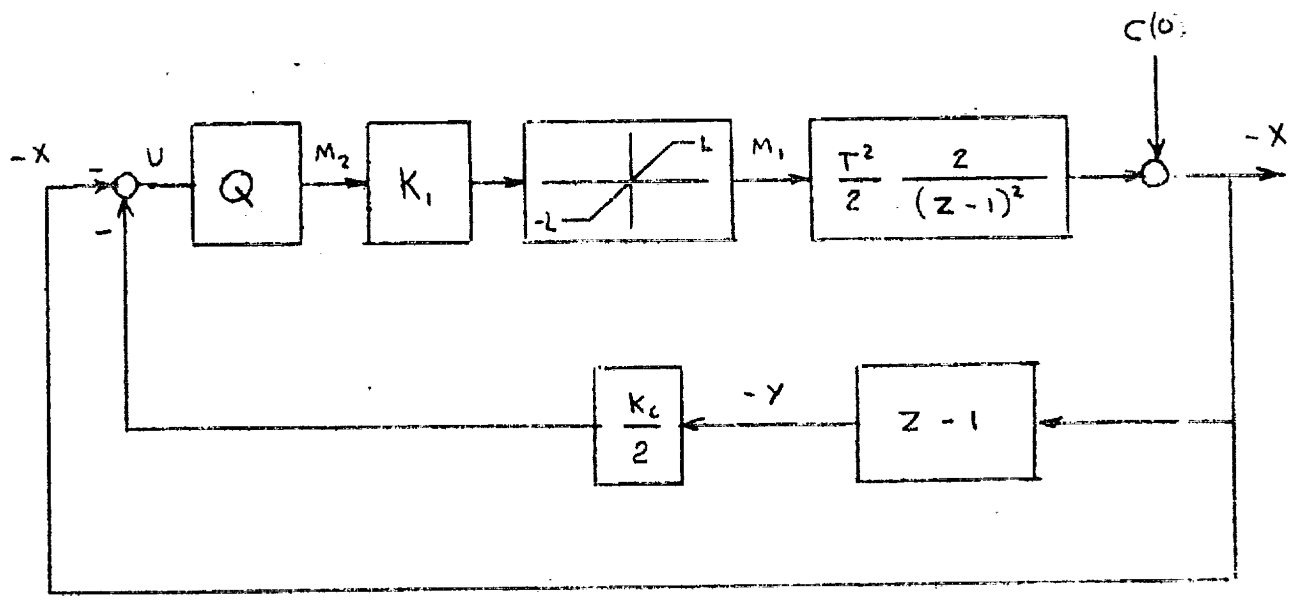

Contactor System Response—Compensated

FIGURE 9




**System With Saturation and Quantization**

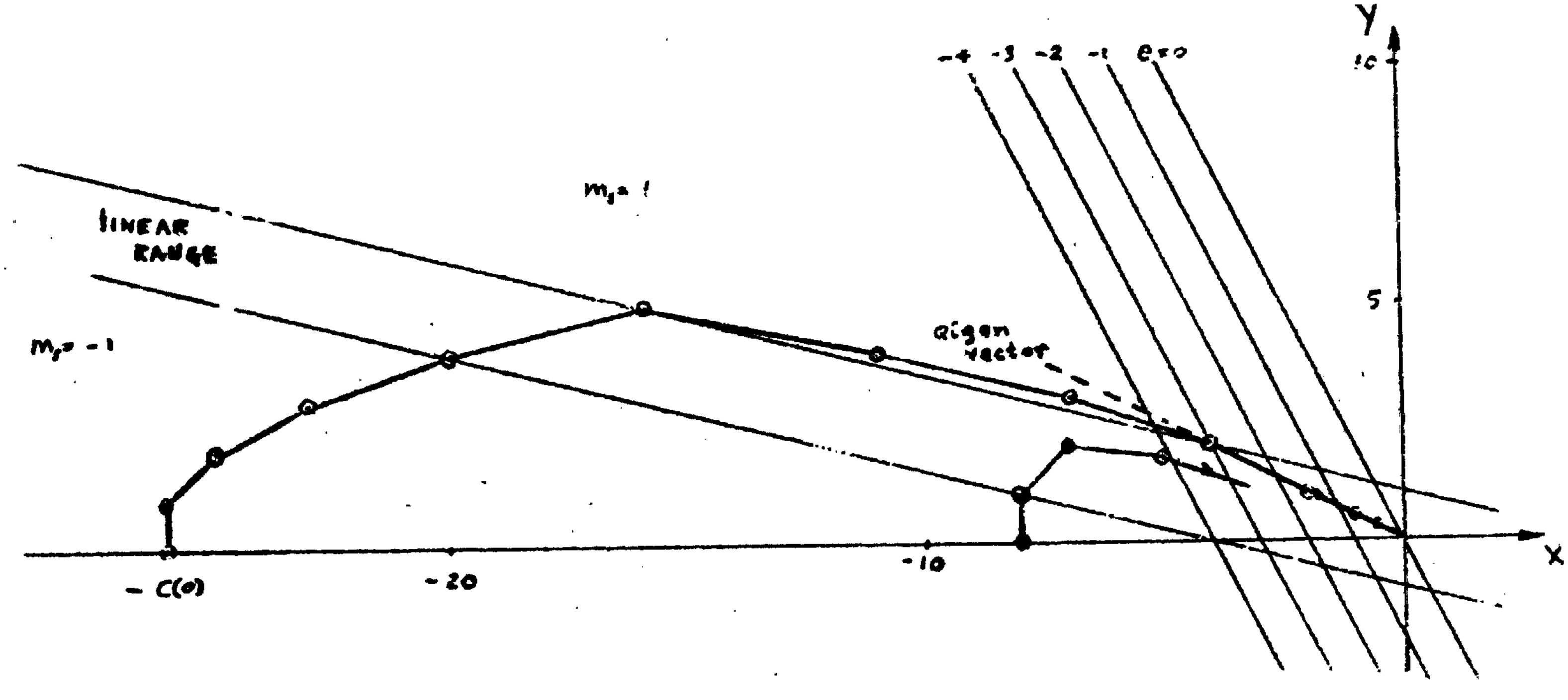
(A)




Transformed System

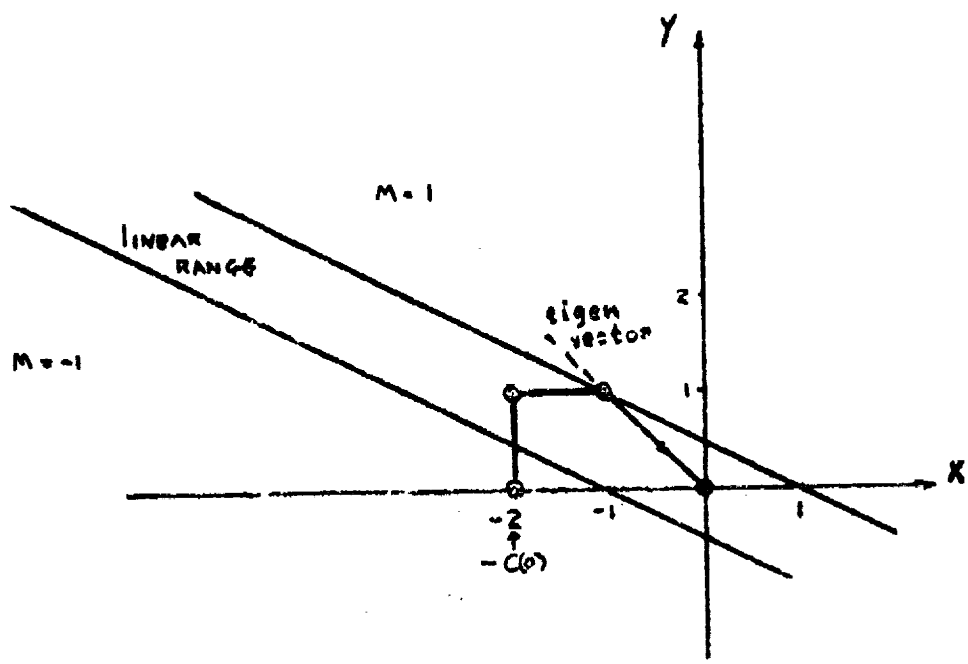
(B)

FIGURE 10



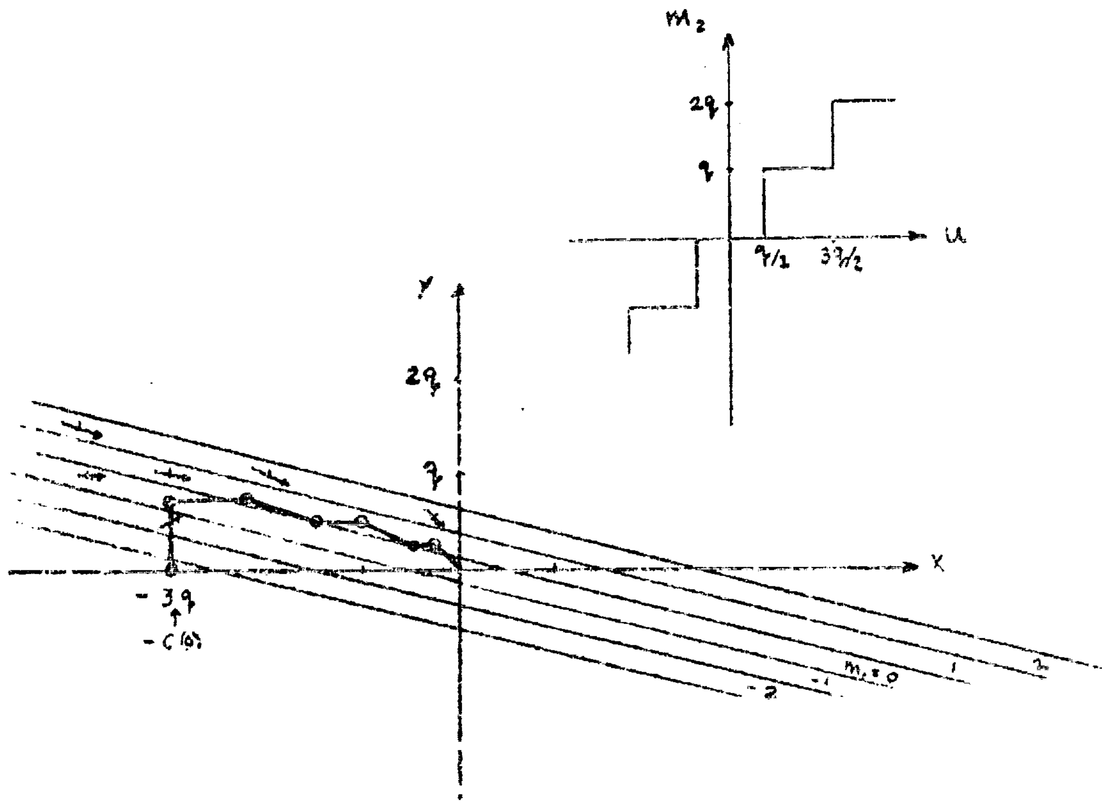

**System with saturation—Small Bandwidth**

FIGURE 11



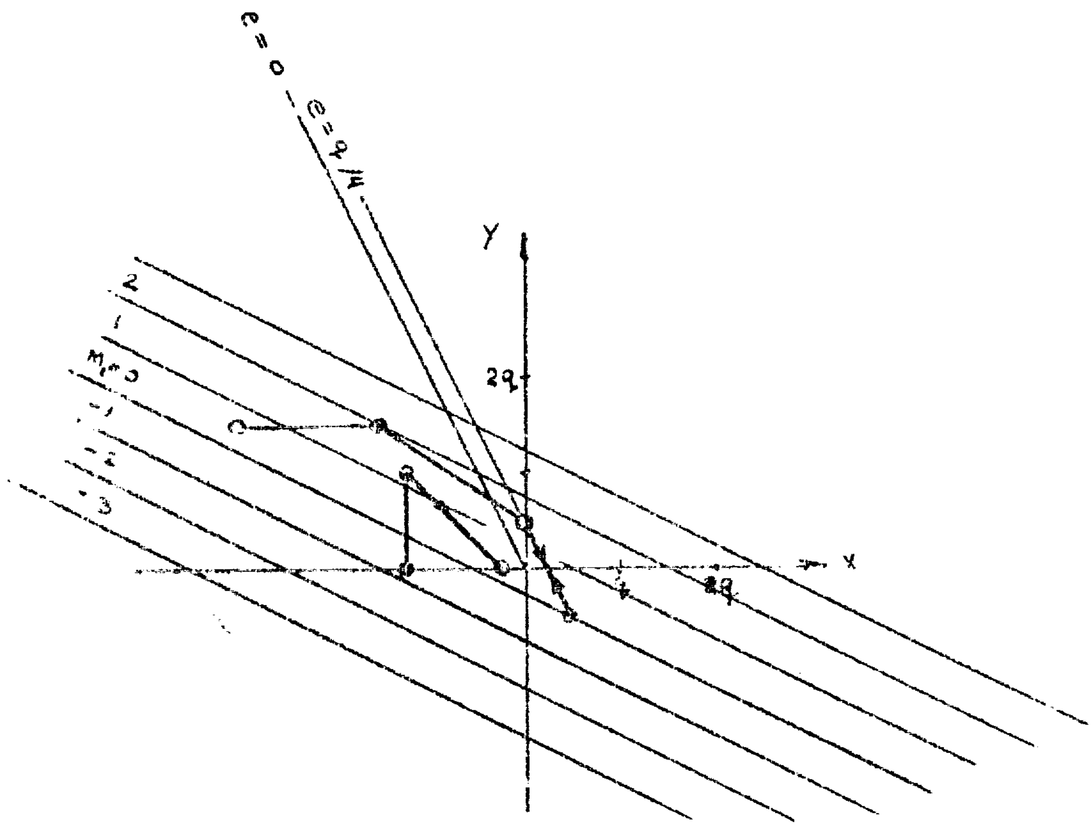

**System with Saturation—Large Bandwidth**

FIGURE 12




Effect of Quantization--Small Bandwidth

FIGURE 13




Effect of Quantization--Large Bandwidth

FIGURE 14